

### Math 524 Exam 11 Solutions

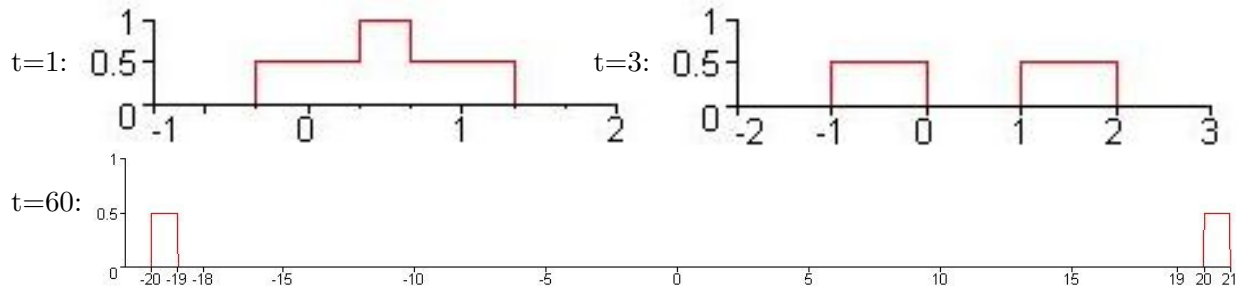
Set  $s(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ . These problems all concern propagation of a square wave, with  $v = 1/3$ , initial position  $f(x, 0) = f_0(x) = s(x)$ , and initial velocity  $\frac{\partial f}{\partial t}(x, 0) = g_0(x) = 0$ . For each of the following problems, compute  $f(x, t)$  for all  $x, t$ , and sketch three solutions: for  $t = 1, t = 3, t = 60$ .

We start with  $h_1(x) = \frac{v f_0(x) - \int g_0(x) dx}{2v}$ ,  $h_2(x) = \frac{v f_0(x) + \int g_0(x) dx}{2v}$ , evaluated at  $f_0(x) = s(x), g_0(x) = 0$ , to get  $\frac{s(x)}{2}$ . Call this  $h(x) = \begin{cases} 1/2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} = h_1(x) = h_2(x)$ .

1. Domain is  $\mathbb{R}$ .

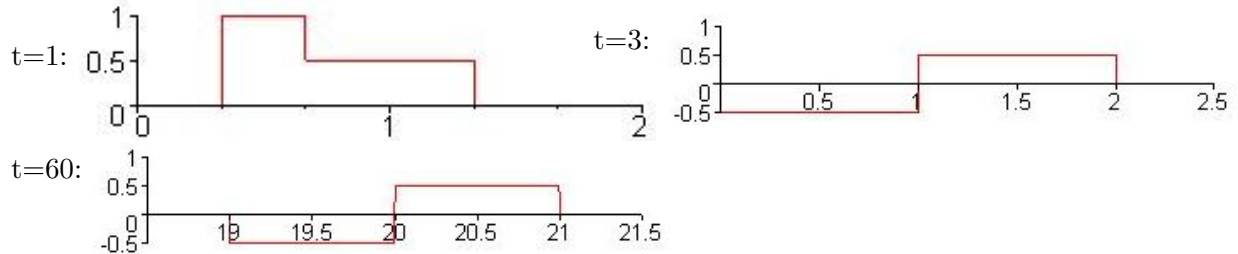
$$f(x, t) = h(x - t/3) + h(x + t/3); \text{ hence } f(x, 1) = h(x - 1/3) + h(x + 1/3) = \begin{cases} 1 & 1/3 < x < 2/3 \\ 1/2 & -1/3 < x \leq 1/3 \\ 1/2 & 2/3 \leq x < 4/3 \\ 0 & \text{otherwise} \end{cases},$$

$$f(x, 3) = h(x - 1) + h(x + 1), f(x, 60) = h(x - 20) + h(x + 20).$$



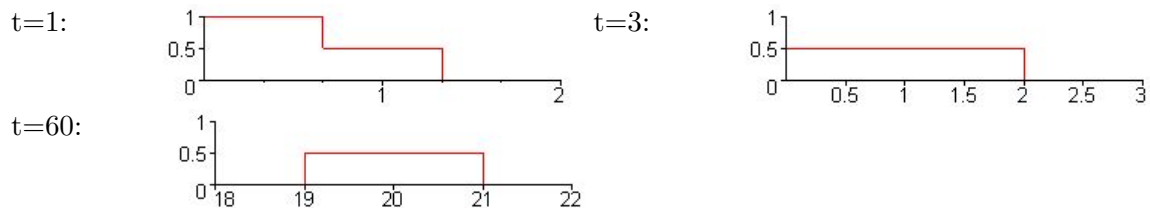
2. Domain is  $[0, \infty)$ , with Dirichlet boundary condition.

$f(x, t) = h(x - t/3) - h(-x - t/3) + h(x + t/3) - h(-x + t/3)$ . The first piece is a positive (above the x-axis) square wave, moving right from  $[0, 1]$ . The second is a negative (below the x-axis) square wave, moving left from  $[-1, 0]$  (it will never enter the domain). The third is a positive square wave, moving left from  $[0, 1]$ . The fourth is a negative square wave, moving right from  $[-1, 0]$ .



3. Domain is  $[0, \infty)$ , with Neumann boundary condition.

$f(x, t) = h(x - t/3) + h(-x - t/3) + h(x + t/3) + h(-x + t/3)$ . This is very similar to the Dirichlet condition, except that all pieces are positive square waves.



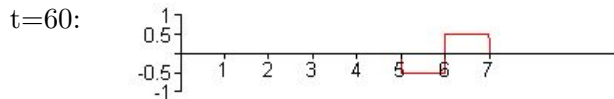
4. Domain is  $[0, 7]$ , with Dirichlet boundary conditions.

$$f(x, t) = \sum_{n=-\infty}^{\infty} \left( h(x-t/3+14n) - h(-x-t/3+14n) + h(x+t/3+14n) - h(-x+t/3+14n) \right)$$

Each of the four square waves has infinitely many copies, all 14 units apart. Because of this large distance, each wave has either 0 or 1 copy on view in  $[0, 7]$  at any time. For  $t = 1, t = 3$ , the waves have not moved very far yet, so the solutions will be identical to problem 2. For  $t = 60$ , however, the waves will have traveled 20 units, which is far enough for the copies to come into play (equivalently, for the waves to bounce off the  $x = 7$  wall).

Mechanical solution: Solving  $0 < x - 20 + 14n < 1$ , we get  $20 - 14n < x < 21 - 14n$ . This is in  $[0, 7]$  only for  $n = 1$ . Solving  $0 < -x - 20 + 14n < 1$ , we get  $-20 + 14n > x > -21 + 14n$ . This is in  $[0, 7]$  for no values of  $n$ . Solving  $0 < x + 20 + 14n < 1$ , we get  $-20 - 14n < x < -19 - 14n$ . This is in  $[0, 7]$  for no values of  $n$ . Solving  $0 < -x + 20 + 14n < 1$ , we get  $20 + 14n > x > 19 + 14n$ . This is in  $[0, 7]$  only for  $n = -1$ . Hence  $f(x, 60) = h(x - 6) - h(-x + 6)$ .

Alternate solution: The rightward-moving wave has its leading edge at  $x = 1$ , and will move 60 time units (equivalently, 20 space units). The first six space units it moves to the right until it hits the right boundary. There, it reflects in direction of movement and flips below the axis. For the next seven space units, it moves to the left until it hits the left boundary. There it again reflects its direction, and flips again to now be above the axis. It then moves  $20 - 7 - 6 = 7$  space steps to the right, ending between 6 and 7, above the axis. The leftmost-moving wave has its leading edge at  $x = 0$ , so in fact it is about to reflect off the left boundary, becoming a negative wave, and move to the right. After 7 space steps it hits the right boundary and reflects, becoming a positive wave. After 7 more space steps it hits the left boundary and reflects, again becoming a negative wave. It then has  $20 - 7 - 7 = 6$  more space steps to the right, ending between 5 and 6.



Note: Neumann boundary conditions would normally be tricky, but because  $\dot{f}(x, 0) = 0$ , in fact it turns out to be just the same as the Dirichlet conditions with both  $-h$  terms turning into  $+h$  terms – the waves remain positive when they reflect off the boundaries.